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VALUING CUSTOMER PORTFOLIOS WITH ENDOGENOUS MASS-AND-DIRECT-MARKETING INTERVENTIONS USING A STOCHASTIC DYNAMIC PROGRAMMING DECOMPOSITION*

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Abstract

Customer Relationship Management generally uses the value of customers to allocate marketing budget. But marketing interventions generally change the customer behavior, turning upside-down the customers ranking based on their initial valuations and making the budget allocation suboptimal. Rational Managers should allocate the marketing budget to maximize the expected net present value of future profits drawn from each customer, simultaneously planning mass marketing interventions and direct marketing effort on each individual. This is a large dimensional Stochastic Dynamic Program, which cannot be easily solved due to the curse of dimensionality. This paper propose a new decomposition algorithm to alleviate the curse of dimensionality in SDP problems, which allows forward-looking firms to allocate the marketing budget optimizing the CLV of their customer base, simultaneously using customized and mass marketing interventions.

Keywords: Marketing Budget allocation, CRM, Stochastic Dynamic Programming.

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1 Introduction

Customers are central assets of the firm, and marketing departments increasingly adopt Customer Relationship Management (CRM) schedules to improve customer acquisition, expenditure and retention. Essentially, CRM involves a systematic allocation of differential resources to customers, based on their individual value to the business. The resources allocated to each customer can be channelled through a mix of alternative interventions, and complemented by mass actions. Traditionally, marketing resource allocation was based on heuristic rules (see Mantrala, 2002). But the benefits of CRM policies are nowadays justified by their impact on firms' return (Rust, Lemon and Zeithaml, 2004). In order to plan the allocation of resources, managers should maximize the value of its customer base. This concept is ideally measured by the summa of Customer Lifetime Values (CLV), that is, the summa of net present values of discounted cash flows between a customer and the firm (Gupta, Lehmann and Stuart 2004, Gupta and Lehmann 2006). The assessment of customers' values, and the effectiveness of a marketing intervention is typically based on the econometric analysis of large customer databases.

CRM requires planning a portfolio of alternative marketing mix interventions. The literature on budget allocation typically considers mass interventions from the marketing mix (advertising promotion and sales force, reference prices and price-promotions, product and production, and distribution channels). For a review see, e.g., Gupta and Steenburgh (2008) and Shankar (2008). The direct marketing literature typically considers a single intervention customized, or at least tailored to small segments. For example, it is common the use of certain pricing decisions (Lewis 2005), catalog mailing (see, e.g., Bitran and Mondschein 1996; Gönül and Shi 1998; Gönül and Ter Hofstede 2006; Simester et al. 2006), couponing decisions (e.g., Bawa and Shoemaker 1987; Rossi et al. 1996), direct mailing (Roberts and Berger 1989) and relationship-oriented magazines (Berry 1995, Bhattacharya and Bolton 1999, McDonald 1998).

Planning the optimal CRM interventions maximizing the global expected CLVs is, by all means, a difficult task. In an attempt to address it, the standard CRM procedure allocates marketing budget to each individual customer, after ranking customers by its CLV value (Reinartz and Kumar 2005, Rust, Lemon and Zeithaml 2004, Venkatesan and Kumar 2004). Assessing new marketing interventions using CLVs computed from historical data is potentially misleading. The planned CRM marketing interventions will change the purchasing behavior of different customers, changing their CLVs, turning upside-down the customers ranking and making our history-based decisions sub-optimal. To cope with this inherent endogeneity, the objective of the allocation marketing models should be a CLV measure computed as the optimal value achieved when the optimal CRM investment is implemented. The idea is that when the CLV is computed we should take into account how customers will react to the changes in the CRM policies.

To avoid this endogeneity problem, some authors have tried to optimize the expected CLVs. Rust and Verhoef (2005) optimize each individual customer's profitability year by year (a myopic planning). Alternatively, other authors optimize the expected CLV using Stochastic Dynamic Programming (SDP). This is a natural approach to solve this problem, but SDP is affected by the curse of dimensionality (the complexity increases drastically with the size of the problems). Therefore, they consider a partial solution, that consists of ignoring mass interventions (aimed to all the customers) focusing on direct individual interventions, so that the investment decision for each customer is independent, and the standard SDP algorithms can be applied to at low computational cost considering "decoupled" decision problem. Gönül and Shi (1998) and Montoya et al. (2007) study direct marketing problems. Khan et al. (2009) estimate the impact of multiple promotional retail instruments, (discount coupons, free shipping offers, and a loyalty program) on customer behavior, designing a customized promotional schedule solving a different SDP problem for each customer. Yet, how to optimize simultaneously both types of interventions (mass, and direct ones) is an unsolved issue, as the SDP optimization problems are not separable among customers. Maximizing the expected CLVs of a customers portfolio with multiple types of personalized and mass marketing interventions, accounting for long term returns, and solving the endogeneity issue is what Rust and Chung (2006, p. 575) called the "Holy Grail" of CRM.

In this paper we provide a fully tailored approach for planning policies that maximize the expected CLV of all the customers in the market accounting for the endogeneity issues. Our approach considers that customer behaviour follows a Markov model in which sales respond to mass and direct marketing interventions, and marketing expenditures are allocated to maximize the summa of expected CLVs for all its customers. Because such models can become rather intractable in general, we propose a method to address this problem by

splitting it into manageable pieces (subproblems) and by coordinating the solutions of these subproblems. With this approach, we obtain two main computational advantages. First, the subproblems are, by definition, smaller than the original problem and therefore much faster to solve. Second, the uncertainty can be easily handled in each subproblems. To validate the efficiency of the approach, we provide a proof of convergence and have solved several stochastic dynamic CLV models. The numerical results show the effectiveness of the method to solve large-scale problems.

We also present an empirical application. We consider a medium size international wholesale company based in eastern Europe of built-in electric appliances for kitchens. This is a firm with various forms of sales response so its marketing budget allocation strategy involves general marketing investments (mainly advertising and promotions in professional fairs) and personalized customer investments. In this research, we therefore investigate whether these two types of interventions differ across customers. The results show that companies should consider different strategies to different customers to achieve long-term profitability over all of the periods of time.

The paper proceeds as follows. In Section 2, we provide a model for dynamically allocating marketing budgets in the context of CRM. The present model considers simultaneously direct marketing interventions tailored to each customer and mass marketing interventions aimed to the customer base. In Section 3, we present the proposed decomposition methodology. In section 4, we illustrate the performance of the algorithm using numerical simulations, and provide a proof of convergence. In Section 5, we present an empirical application to customers of manufacturer of electric appliances. Finally, in Section 6, we discuss the results and provide some concluding remarks. The Appendix provides technical details about the algorithm implementation.

2 A Model for optimal dynamic budget allocation in CRM

Planning marketing interventions in CRM requires managers to allocate budget dynamically maximizing the summa of expected CLVs from all customers, based on historic customer state information. To address the optimal budget allocation problem, the firm must carry out two tasks (see, e.g., Gupta et al. 2009):

- Task 1. Estimate the expected CLV building analytical models to forecast future sales response by customers (Gupta and Lehmann 2003, 2005, Kamakura et al. 2005, Gupta and Zeithaml 2006); and
- Task 2. Solve the stochastic dynamic optimization problem including all individual customers (see, e.g., Rust and Verhoef 2005, Rust and Chung 2006).

The first task requires the design of a dynamic panel sales response model. Let $\mathbb{I} = \{1, \dots, I\}$ be a finite set of active customers and $t \in \{0, 1, 2, \dots\}$ the time index. The firm chooses a sequence of dynamic controls:

- e_{it} is the direct marketing interventions on customer $i \in \mathbb{I}$ at period of time t , such as personalized advertising and directed promotional expenditures. We use the notation $e_t = (e'_{1t}, \dots, e'_{It})'$, where e' denotes the transpose of e .
- A_t is the mass marketing interventions at period of time t ,
- P_t denotes the prices for the different products.

These controls (A_t, P_t, e_t) are defined on the a control set \mathcal{A} , a Borel-measurable subset of the Euclidean space.

The dynamic control variables have an effect on the customer behavior state variables. We will consider the following state model:

- S_{it} is the random vector describing the sales-level state of customer $i \in \mathbb{I}$ at time t , and we use the notation $S_t = (S_{1t}, \dots, S_{It})'$. With probability one, S_t takes values on a set of states \mathcal{S} a Borel-measurable subset of the Euclidean space.
- We assume that S_t follows a Markovian process with transition probability

$$F(s'|s, A, P, e) = \Pr(S_t \leq s' | S_{t-1} = s, A_{t-1} = A, P_{t-1} = P, e_{t-1} = e) = \prod_{i \in \mathbb{I}} F_i(s'_i | s_i, A, P, e_i)$$

The typical example is when the company considers a dynamic panel model where each customer satisfies

$$S_{it} = \rho S_{it-1} + g_i(A_{t-1}, P_{t-1}, e_{it-1}) + \varepsilon_{it} \quad (1)$$

where $|\rho| < 1$, the innovation ε_{it} is a strong white noise independent for each customer with cumulative distribution $H_i(\cdot)$. The functions $g_i(\cdot)$ and $H_i(\cdot)$ are continuous and can vary across customers to allow heterogeneity in the expected responses, so that

$$F_i(s'_i | s_i, A, P, e_i) = \Pr\{\varepsilon_i \leq s'_i - \rho s_i - g_i(A, P, e_i)\} = H_i(s'_i - \rho s_i - g_i(A, P, e_i)).$$

The one-lag memory structure imposed by the Markov dependence assumption can be relaxed by considering p-lags autoregressive models in the space-of-states.

The dynamic model can be estimated using standard econometric techniques for time series cross-section and/or dynamic panels. Firms increasingly store large panel data basis with information about their customers, including social information (such as socio-demographic, geographic information, lifestyle habits) and trade internal data (such as historical transaction records, customers feedback, or Web browsing records), see Bose and Chen (2009). The econometric literature has developed a battery of linear and nonlinear models for the dynamic analysis of large data-panels, and the marketing researchers have tailored these models for the prediction of future purchases at customer-level (e.g., Schmittlein and Peterson 1994). Using these tools, company managers often estimate the expected CLV for each customer based on its past behavior, (generally in a *ceteris paribus* context, omitting or fixing the marketing mix variables).

The main contribution of this paper is to propose a methodology for solving Task 2. The firm should choose the CRM policy maximizing the expected sum of its CLVs, constrained to the customer response to feasible marketing policies. This problem is a large dimensional (discounted) SDP problem. In other words, we consider that a rational forward-looking firm has to decide on CRM budget allocation policies over time, drawing profits

$$r(S_t, A_t, P_t, e_t) := \sum_{i \in \mathbb{I}} r_i(S_{it}, A_t, P_t, e_{it}) \quad (2)$$

at each period of time $t > 0$ from all of their customers¹. Let $\delta \in (0, 1)$ be a time discount parameter, then we assume that the company maximizes the expected net present value $E_0 \left[\sum_{t \geq 0} \delta^t r(S_t, A_t, P_t, e_t) \right]$.

Marketing budget decisions generally face corporate constraints settled by the interactions between managers, bond holders, and stockholders. We consider that for each state S_{t-1} , there is a non-empty compact set $\mathbb{A}(S_{t-1}) \subset \mathcal{A}$ of admissible controls at time $t > 0$ which depends upon the previous period sales; i.e. $(A_t, P_t, e_t) \in \mathbb{A}(S_{t-1})$. The admissible state-controls pairs are given by $\mathcal{K} := \{(S, A, P, e) : S \in \mathcal{S}, (A, P, e) \in \mathbb{A}(S)\}$. As usual, we assume that $|r(S, A, P, e)|$ is bounded on \mathcal{K} except for a null probability set.

Problem 1 *Given the initial state S_0 , the firm faces the following problem:*

$$\max_{\{(A_t, P_t, e_t) \in \mathbb{A}(S_{t-1})\}_{t \geq 0}} E_0 \left[\sum_{t \geq 0} \delta^t r(S_t, A_t, P_t, e_t) \right] := V(S_0)$$

As usual, we denote the maximum $V(S_0)$ as the “value function”.

This is a SDP problem in discrete time. Problem 1 is solved by the optimal policy $(A^*(s), P^*(s), e^*(s))$, which is a time-invariant function prescribing the best decision for each state s , i.e.

$$V(S_0) = E_0 \left[\sum_{t \geq 0} \delta^t r(S_t, A^*(S_{t-1}), P^*(S_{t-1}), e^*(S_{t-1})) \right].$$

Interestingly, for each period of time t , we can interpret $V(S_t)$ as the expected present discounted value of profits under the current state S_t . Under certain regularity conditions, the optimal policy function

¹We use the standard notation “:=” for definitions.

$(A^*(s), P^*(s), e^*(s))$ are characterized by the value function $V(\cdot)$ as the solution of the Jacobi-Bellman equation (Bellman, 1955, 1957):

$$V(S_t) = \max_{(A_t, P_t, e_t) \in \mathbb{A}(S_{t-1})} \{r(S_t, A_t, P_t, e_t) + \delta E_t[V(S_{t+1})]\}.$$

Given the optimal policy rule, managers can make optimal decisions on marketing activities $(A^*(s), P^*(s), e^*(s))$, that maximize their expected profits given the sales state s observed in the previous period. Also, $V(S_t)$ gives us the company value derived from the CLVs customer portfolio at time t , provided that the firms are optimally managed. Using optimal policies for solving the SDP problem has several advantages: they are Simple (ease of understanding for managers) and Adaptive (the decisions can be automatically updated as new state-information becomes available). Note that they can be used also for simulation. For each period of time t , given S_t drawn from the conditional distribution $F(s|S_{t-1}, A_{t-1}, P_{t-1}, e_{t-1})$, the values $A_{t+1} = A^*(S_t)$, $P_{t+1} = P^*(S_t)$, $e_{t+1} = e^*(S_t)$ can be used to simulate Monte Carlo scenarios, and then to compute numerically the expected path for the optimal policies $E[A_t]$, $E[P_t]$, $E[e_t]$ and states $E[S_t]$, as well as confidence intervals.

The computation of large SDP remains one of the most challenging optimization problem. Most problems can become intractable as the dimension of the state space increases (the CPU time to calculate a value function increases exponentially in the dimension of the state space), which is the well known "curse of dimensionality" (Bellman, 1961). Due to the curse of dimensionality, SDP problems can be solved numerically for decision problems in which only few state variables are considered. This implies that CRM decision problems with more than 3 customers cannot be solved using the standard approaches: value iteration and policy iteration (see Appendix A for an introduction).

One of the classical strategies to solve large decision problems are the decomposition based approaches. There exists several mathematical programming decomposition algorithms for large optimization problems with an appropriate structure (Danzting-Wolfe and Benders-decomposition in convex problems, and augmented Lagrangian relaxation in nonconvex problems). Some attempts to solve large SDP problems combine traditional decomposition algorithms and statistical sampling techniques. Sampling is used to create a scenario tree that represents the uncertainty (Heitsch and Römisch, 2009). Then the original problem is approximated by a finite deterministic one. The dimension of the tree grows exponentially with the number of states variables, and so does the complexity of the deterministic problem. To tackle this issue, a decomposition method is used such as Benders and Lagrangian schemes (see Birge and Louveaux, 1997), but these methods may converge slowly in practice (see Chun and S.M. Robinson, 1995). In contrast, the current paper first considers the decomposition of the original stochastic problem using the law of iterated expectations, and then, each subproblem is solved either using value-iteration or policy-iteration algorithms. It must be noted that this approach represents a general and versatile tool, as it describes how marketing policies evolve over an infinite number of time periods, and the expected present value of those decisions.

3 Solving the SDP using a Bellman-decomposition algorithm

In this section we present the decomposition approach to address large CRM problems. To attain this goal, we first assume,

Condition 2 *There is a random vector $\bar{S}_t := h(S_t)$ where $h(\cdot)$ is a measurable function from the state space to another Euclidean space of low dimension, such that the expected effect of S_t on $r(S_t, A_t, P_t, e_t)$ can be summarized in the index \bar{S}_t , i.e.*

$$E_0[r(S_t, A_t, P_t, e_t) | S_t, A_t, P_t, e_t] = E_0[r(S_t, A_t, P_t, e_t) | \bar{S}_t, A_t, P_t, e_t], \quad a.e. \quad (3)$$

A relevant example in which this condition is satisfied, is the decision problems in which managers' objectives are given by:

$$\begin{aligned} r(S_t, A_t, P_t, e_t) & : = (P_t - c_0) I \bar{S}_t - \sum_{i \in \mathbb{I}} c_i(e_{it}) - c_m(A_t), \\ r_i(S_{it}, A_t, P_t, e_{it}) & : = (P_t - c_0) \cdot S_{it} - c_i(e_{it}) - c_m(A_t) / I, \end{aligned} \quad (4)$$

i.e., they optimize the value drawn from a measurement of total sales $\sum_{i \in \mathbb{I}} S_{it} = I \bar{S}_t$, where $(P_t - c_0)$ is the unit margin, $c_m(\cdot) \geq 0$ is the cost of mass advertising interventions (when A_t are monetary units, c_m is the identity function) and $c_i(\cdot) \geq 0$ is the cost of the direct marketing interventions on customer i .

Next, we discuss the transition of the index $\bar{S}_t = h(S_t)$, given by

$$F(\bar{s}' | \bar{s}, A, P, e) = \Pr(\bar{S}_t \leq \bar{s}' | \bar{S}_t = \bar{s}, A, P, e) = \int_{\{h(s) \leq \bar{s}'\}} F(ds' | \bar{s}, A, P, e),$$

where $\bar{s} = h(s)$ and $F(s' | \bar{s}, A, P, e) = E[F(s' | S_t, A, P, e) | h(S_t) = \bar{s}, A, P, e]$. In practice, the computation of $F(\bar{s}' | \bar{s}, A, P, e)$ may require the use of numerical methods, but the analysis is particularly simple when we consider dynamic panels as described in (1), and $\bar{S}_t = I^{-1} \sum_{i \in \mathbb{I}} S_{it}$ as in (4), using that

$$\bar{S}_t = \rho \bar{S}_{t-1} + \bar{g}(A_{t-1}, P_{t-1}, e_{t-1}) + \bar{\varepsilon}_t,$$

where $\bar{\varepsilon}_t = I^{-1} \sum_{i \in \mathbb{I}} \varepsilon_{it}$ has probability distribution $G_I(\bar{\varepsilon}) = G^*(\bar{\varepsilon}/I)$ with $G^* = G_1 * \dots * G_I$ the convolution of individual shocks' distributions, and $\bar{g}(A, P, e) = \sum_{i \in \mathbb{I}} g_i(A, P, e_i)/I$, so that

$$F(\bar{s}' | \bar{s}, A, P, e) = G_I(\bar{s}' - \rho \bar{s} - \bar{g}(A, P, e)).$$

Finally we assume that admissible prices, mass and direct marketing interventions are bounded by a maximum level which can be adapted to the previous state of sales.

Condition 3 *The non-empty compact set $\mathbb{A}(S) \subset \mathcal{A}$ is defined for all S as*

$$\begin{aligned} \mathbb{A}(S) := & \left\{ (A, P, e) \in \mathcal{A} : A^l_{(\bar{S})} \leq A \leq A^u_{(\bar{S})}, \right. \\ & \left. P^l_{(\bar{S})} \leq P \leq P^u_{(\bar{S})}, e_i^l(S_i) \leq e_i \leq e_i^u(S_i) \right\}, \end{aligned}$$

S_i is the i -th coordinate of S , and where $\bar{S} = h(S)$ and $0 \leq A^l \leq A^u$, $0 \leq P^l \leq P^u$, $0 \leq e_i^l \leq e_i^u$ are bounded continuous functions in \mathcal{S} .

Let us define the subproblems:

$$\begin{aligned} V_i(s_i) & : = \max_{\{e_{it}\}} E_0 \left[\sum_{t \geq 0} \delta^t R_i(S_{it}, e_{it}) \right], \quad \text{for all } i \in \mathbb{I}, \\ \bar{V}(\bar{s}) & = \max_{\{A_t, P_t\}} E_0 \left[\sum_{t \geq 0} \delta^t R(\bar{S}_t, A_t, P_t) \right], \end{aligned}$$

where $R_i(S_{it}, e_{it})$ and $R(\bar{S}_t, A_t, P_t)$ are conditional expectations

$$\begin{aligned} R_i(S_{it}, e_{it}) &= I \cdot E[r_i(S_{it}, A_t^*, P_t^*, e_{it}) | S_{it}, e_{it}], \\ R(\bar{S}_t, A_t, P_t) &:= E[r(S_t, A_t, P_t, e_t^*) | \bar{S}_t, A_t, P_t], \end{aligned} \tag{5}$$

with A_t^* , P_t^* , e_t^* the optimal decisions for time t .

Notice that any policy function (A, P, e) , by the *Law of Iterated Expectations* it is satisfied that

$$\begin{aligned} E_0 \left[\sum_{t \geq 0} \sum_{i \in \mathbb{I}} \delta^t r_i(S_{it}, A_t, P_t, e_{it}) \right] &= \sum_{i \in \mathbb{I}} E_0 \left[\sum_{t \geq 0} \delta^t E[r_i(S_{it}, A_t, P_t, e_{it}) | S_{it}, e_{it}] \right] \\ &= E_0 \left[\sum_{t \geq 0} \delta^t E \left[\sum_{i \in \mathbb{I}} r_i(S_{it}, A_t, P_t, e_{it}) | \bar{S}_t, A_t, P_t \right] \right], \end{aligned}$$

where $A_t = A(S_{t-1})$, $P_t = P(S_{t-1})$, $e_t = e(S_{t-1})$; which under conditions (2) and (3) imply that $V(s) = I^{-1} \sum_{i \in \mathbb{I}} V_i(s_i)$ and also that $V(s) = \bar{V}(\bar{s})$ almost everywhere.

Therefore the subproblems $\{V_i(s_i)\}_{i \in \mathbb{I}}$ and $\bar{V}(\bar{s})$ characterize the value function $V(\cdot)$, the subproblems are, by definition, smaller than the original problem (Problem 1) and therefore much faster to solve. In order to solve the subproblems separately, we need the transition kernel for $\{V_i(s_i) : i \in \mathbb{I}\}$ and $\bar{V}(\bar{s})$ respectively given by

$$\begin{aligned}\mathbb{F}_i(s'_i | s_i, e_i) &= E[F_i(s_i | S_{it-1}, A_{t-1}^*, P_{t-1}^*, e_{it-1}) | S_{it-1} = s_i, e_{it-1} = e_i], \quad \text{for all } i \in \mathbb{I}, \\ \mathbb{F}(\bar{s}' | \bar{s}, A, P) &= E[F(\bar{s}' | \bar{S}_{t-1}, A_{t-1}, P_{t-1}, e_{t-1}^*) | \bar{S}_{t-1} = \bar{s}, A_{t-1} = A, P_{t-1} = P].\end{aligned}$$

and we need also to know $R_i(S_{it}, e_{it})$ and $R(\bar{S}_t, A_t, P_t)$. The computation of the required conditional probabilities and expectations is unfeasible since the optimal policy function (A^*, P^*, e^*) is unknown.

3.1 The algorithm

The general scheme of the algorithm is stated as follows.

ALGORITHM

1. **Initialization:** Choose a scenario set of states and a starting policy $\{A^k(\bar{s}), P^k(\bar{s}), e^k(s)\}$ with $e^k(s) = (e_1^k(s_1), \dots, e_I^k(s_I))$. Set $k = 0$.

2. **Repeat:**

- 2.1 Generate recursively $\{S_t^k, A_t^k, P_t^k, e_t^k\}_{t=1}^T$ where S_t^k is drawn from

$$F\left(s | S_{t-1}^k, A^k(\bar{S}_{t-1}^k), P^k(\bar{S}_{t-1}^k), e^k(S_{t-1}^k)\right),$$

and compute $\bar{S}_t^k = h(S_t^k)$,

- 2.2. With the simulated data compute

$$\begin{aligned}R_i^k(S_{it}, e_{it}) &= I \cdot E[r_i(S_{it}, P_t^k, A_t^k, e_{it}) | S_{it}, e_{it}], \\ R^k(\bar{S}_t, A_t, P_t) &= E[r(S_t, P_t, A_t, e_t^k) | \bar{S}_t, A_t, P_t].\end{aligned}$$

and the kernels

$$\begin{aligned}\mathbb{F}^k(s'_i | s_i, e_i) &= \Pr(S_{it}^k \leq s'_i | S_{it-1}^k = s_i, e_{it-1} = e_i), \quad i \in \mathbb{I}, \\ \mathbb{F}^k(\bar{s}' | \bar{s}, A, P) &= \Pr(\bar{S}_t^k \leq \bar{s}' | \bar{S}_{t-1}^k = \bar{s}, A_{t-1} = A, P_{t-1} = P).\end{aligned}$$

- 2.3 Solve the SDP subproblems

$$\max_{\{e_{it} \in \mathbb{A}_i(S_{it-1})\}_{t \geq 0}} E \left[\sum_{t \geq 0} \delta^t R_i^k(S_{it}, e_{it}) | S_{i0} = s_i \right] := V_i^k(s_i),$$

in $\{e_{it}^k\}_{t \geq 0}$ for each $i \in \mathbb{I}$, where $\mathbb{A}_i(S_{it-1}) = \{e_i : 0 \leq e_i \leq \bar{e}_i(S_{it-1})\}$.

- 2.4 Solve the SDP subproblem

$$\max_{\{A_t, P_t\} \in \bar{\mathbb{A}}(\bar{S}_{t-1})} E \left[\sum_{t \geq 0} \delta^t R^k(\bar{S}_t, A_t, P_t) | \bar{S}_0 = \bar{s} \right] := \bar{V}^k(\bar{s}),$$

where

$$\bar{\mathbb{A}}(\bar{S}_{t-1}) = \{(p, A) : 0 \leq A \leq \bar{A}(\bar{S}_{t-1}), 0 \leq P \leq \bar{P}(\bar{S}_{t-1})\}.$$

- 2.5 Update $\{e_{it}^k, A_t^k, P_t^k\}$ to $\{e_{it}^{k+1}, A_t^{k+1}, P_t^{k+1}\}$, and set $k \leftarrow k + 1$.

3. **Until convergence:** for some tolerance $\epsilon > 0$, when the stopping criteria are satisfied

- Criterion 1: $\max \left\{ \sup_t \frac{|A_t^{k+1} - A_t^k|}{1 + \|A_t^k\|_\infty}, \sup_t \frac{|P_t^{k+1} - P_t^k|}{1 + \|P_t^k\|_\infty}, \sup_{t,i} \frac{|e_{it}^{k+1} - e_{it}^k|}{1 + \|e_{it}^k\|_\infty} \right\} < \epsilon,$
- Criterion 2: $\sup_{S_0} \left\{ \frac{|I^{-1} \sum_{i \in \mathbb{I}} V_i^{k+1}(S_{i0}) - \bar{V}^{k+1}(\bar{S}_0)|}{1 + \|I^{-1} \sum_{i \in \mathbb{I}} V_i^{k+1}(S_{i0})\|_\infty} : \bar{S}_0 = I^{-1} \sum_{i \in \mathbb{I}} S_{i0} \right\} < \epsilon,$

where the superscript k denotes the current iteration and $\|\cdot\|_\infty$ is the supremum norm.

The algorithm iterates the solution of both types of subproblems. For one set of subproblems, the decision variables are only the direct marketing intervention $\{e_{it}\}_{t>0}$. Once the solutions for these subproblems have been computed, price and mass marketing intervention $\{P_t, A_t\}_{t>0}$ are updated. An economic interpretation of the decomposition draws on this partition of the decision variables into individual and general decisions taken among customers. The convergence of the algorithm is discussed in Appendix B.

Any classical method to solve SDP such as value iteration or policy iteration can be applied in steps 2.3 and 2.4, since the subproblems are small problems with just one state variable, using as initial point the optimal policy computed in the previous iteration of the algorithm. The specific details are described in Appendix C.

Note that the value function for the original problem $V(S_1, \dots, S_I)$ and the associated policy functions $[A, P, e](S_1, \dots, S_I)$ cannot be graphically represented for more than two customers due to the dimension. However, graphical figures for these functions would be intuitive user-friendly tools for marketing managers. Interestingly, our algorithm overcomes this problem providing useful and visual tools for managers implementing CRM. After convergence of the algorithm at step k^* to a numerical solution of the original problem, we can depict graphically in the plane the reduced value function $V^{k^*}(\bar{S})$ and the associated reduced optimal policy functions $A^{k^*}(\bar{S}), P^{k^*}(\bar{S})$ to provide graphical rules for planning optimally mass advertising and price (provided that the optimal individual e is implemented). Furthermore, we can depict in the plane the reduced value function $V_i^{k^*}(S_i)$ and the associated reduced optimal policy function $e_i^{k^*}(S_i)$ for the i -th customer, which provide a graphical rule for planning optimally the marketing effort on i -individual (provided that the optimal mass advertising and price have been implemented as well as the effort on other individuals).

4 Some numerical simulations

Let us consider a dynamic-regression model where sales follow a dynamic panel model

$$S_{it} = \rho S_{it-1} + \alpha_{1i} + \beta_{1i} e_{it-1} + \beta_{2i} A_{t-1} + \beta_{3i} P_{t-1}^{\beta_{4i}} + \varepsilon_{it}$$

with $\{\beta_{1i}, \beta_{2i}\} > 0$, and $|\rho| < 1$, where $\{e_{it}\}_{t \geq 1}$ are individual marketing efforts, $\{A_t\}_{t \geq 1}$ is the mass marketing effort, $\{P_t\}_{t \geq 1}$ is the price, and $\{\varepsilon_{it}\}_{t \geq 1}$ are independent white noise processes $N(0, \sigma I)$. We assume that $\{e_{it}\}_{t \geq 1}$ and $\{A_t\}_{t \geq 1}$ are given by a cost function $c(x) = \gamma x^\phi$, with $\gamma > 0$. Then, given $\delta \in (0, 1)$, the firm aims to maximize the expected net present value $E_0 \left[\sum_{t \geq 0} \delta^t r(S_t, A_t, P_t, e_t) \right]$ with

$$r(S_t, A_t, P_t, e_t) := (P_t - c_0) \sum_{i \in \mathbb{I}} S_{it} - c(A_t) - \sum_{i \in \mathbb{I}} c(e_{it}).$$

We have implemented our decomposition algorithm using MATLAB 7.6 on an Intel Core vPro i7 with machine precision 10^{-16} . The algorithm stops whenever $\epsilon = 10^{-8}$.

First, we consider a simplified model in which prices are considered as given, i.e. $\beta_{3i} = 0$ and using a constant exogenous margin m_0 instead of $(P_t - c_0)$. For $m_0 = 50$, $\rho = 0.2$, $\alpha_i = 60$, $\beta_{1i} = 1.2$, $\beta_{2i} = 1.2$, $\sigma = 5$, Table 1 reports the running times (in seconds) until convergence considering different number of customers I , and both policy iteration and value iteration algorithms to solve Steps 2.3 and 2.4 of the algorithm.

Table 1: Properties of the algorithm for different problem sizes in a model without prices.

Method	Number of Customers	Stopping Criteria		Number of Iterations	Computational Time (in seconds)
		Criterion 1	Criterion 2		
Policy Iteration	1	0.0000	0.0000	3	3.8922
	5	0.0000	0.0007	4	11.0790
	25	0.0000	0.0008	4	60.5070
	50	0.0000	0.0009	4	166.9300
	100	0.0000	0.0009	4	687.4300
Value Iteration	1	0.0000	0.0000	4	3.5335
	5	0.0000	0.0007	3	8.6160
	25	0.0000	0.0008	4	61.1350
	50	0.0000	0.0009	4	169.0700
	100	0.0000	0.0009	3	545.9600

Then, we extend the basic model to the general case in which prices are considered as a decision variable. For $c_0 = 50$, $\rho = 0.2$, $\alpha_i = 60$, $\beta_{1i} = 1.2$, $\beta_{2i} = 1.2$, $\beta_{4i} = -0.5$, $\beta_{3i} = 0.5$, $\sigma = 5$, Table 2 reports the running times (in seconds) until convergence considering different number of customers I . The results show that the proposed algorithm is capable of solving the problem with many customers in a reasonable amount of computer time.

Table 2: Properties of the algorithm for different problem sizes in a model with prices.

Method	Number of Customers	Stopping Criteria		Number of Iterations	Computational Time (in seconds)
		Criterion 1	Criterion 2		
Policy Iteration	1	0.0000	0.0527	3	5.9819
	5	0.0000	0.0202	4	29.5548
	25	0.0000	0.0202	4	150.4374
	50	0.0000	0.0202	4	404.8369
	100	0.0000	0.0202	2	873.8870
Value Iteration	1	0.0000	0.0115	6	11.2360
	5	0.0000	0.0202	2	16.4847
	25	0.0000	0.0202	2	83.7036
	50	0.0000	0.0324	2	189.4250
	100	0.0000	0.0202	2	663.1727

These results suggest that the proposed methodology is an effective and useful tool for solving this type of problems as it breaks down a high-dimensional problem into many low-dimensional ones, hence reducing the curse of dimensionality. It is remarkable that the standard policy iteration approach cannot solve a problem of more than 3 customers.

5 An empirical application of a manufacturer of kitchen appliances

In this section we provide an application of the method. We consider a medium size international wholesale company based in eastern Europe. This company distributes and also manufactures a large range of built-in electric appliances for kitchens (such as cookers, ovens and hobs, cooker and chimney hoods, external motors, microwaves, dishwashers, washing machines, refrigerators, and related accessories). The company invests in general marketing effort (mainly advertising and promotions in professional fairs) and personalized investments in their customer relationships management. We do not provide additional company information by confidentiality requests of the company managers.

We use a monthly customer-panel from this company spanning from January 2005 to December 2008. The panel is unbalance, although the vast majority of the clients purchases practically every month within the sample period. As the company sells a wide range of products with different sales to each client, they aggregated their data providing us the monthly net-profit drawn from each client. Therefore, in this section $Y_{i,t}$ is regarded as the financial value obtained from client i at time t , the individual marketing effort on this customer is denoted by $e_{i,t}$, and the general marketing effort is A_t . The basic Markovian model is a dynamic-panel specification

$$Y_{i,t} = \rho Y_{i,t-1} + \beta_1 \ln A_{t-1} + \beta_2 \ln e_{i,t-1} + (\eta_i + u_{it}), \quad E[u_{i,t}X_{i,t}] = 0, \quad E[u_{i,t}] = 0,$$

for all i, t , where $|\rho| < 1$, $u_{i,t}$ is white noise and η_i is a zero mean random coefficient accounting for individual heterogeneity in customer profitability levels. The noise $v_{it} = \eta_i + u_{i,t}$ is autocorrelated due to the stability of η_i , and therefore the OLS and the Within-Group estimators are both inconsistent (as $Y_{i,t-1}$ is a regressor). Taking first differences in the model, we eliminate the specific group effects

$$\Delta Y_{i,t} = \rho \Delta Y_{i,t-1} + \Delta X'_{i,t-1} \beta + \Delta u_{i,t}, \quad t = 2, \dots, T,$$

where $X'_{i,t-1} = (\ln A_{t-1}, \ln e_{i,t-1})'$. The errors $\{\Delta u_{it}\}$ are no longer independent but follow a non invertible $MA(1)$. This equation can be estimated by Instrumental Variables (IV), as proposed by Andersen and Hsiao (1982). It is convenient to use lags of the variable in levels $Y_{i,t-1}$ as instrument, as well as lags of other exogenous regressors. Nonetheless, the IV estimator is not efficient due to the fact that only a few moment conditions are used. Arellano and Bond (1991) proposed a GMM estimator dealing with this problem. The Arellano and Bond (1991) estimators can perform poorly in certain cases, and the method was refined by Blundell and Bond (1998) who included additional moment conditions (building on previous work by Arellano and Bover, 1995). The model was estimated in STATA using the Blundell-Bond refinement. Table 2 reports the estimators of this model. The Wald global significance test is 169.73 distributed as a χ^2_3 with a p-value 0.0000.

Table 2: Main coefficients in the dynamic-panel model for customer profitability.

Y_{t-1}	Coef.	Std. Err	z	P> z
$Y_{i,t-1}$.024	0.011	2.15	0.031
A_{t-1}	821.52	235.244	3.49	0.000
$e_{i,t-1}$	1175.05	172.395	6.82	0.000

In order to improve the heterogeneity analysis, we have decided to include additional information, classifying clients by continental location (4 large regions with dummies $\{D_{ki}\}_{k=1}^4$), and a customers' strategic classification by the company (3 levels with dummies $\{d_{ji}\}_{j=1}^3$), so that we have 12 basic segments. Therefore, we introduce heterogeneity in the response to marketing effort as

$$\begin{aligned} Y_{i,t} = & \rho Y_{i,t-1} + \beta_1 \ln A_{t-1} + \beta_2 \ln e_{i,t-1} + \\ & + \sum_{j=1}^3 \gamma_j (d_{ji} \times \ln A_{t-1}) + \sum_{j=1}^3 \gamma'_j (d_{ji} \times \ln e_{i,t-1}) \\ & + \sum_{k=1}^4 \alpha_k (D_{ki} \times \ln A_{t-1}) + \sum_{k=1}^4 \alpha'_k (D_{ki} \times \ln e_{i,t-1}) + (\eta_i + u_{it}). \end{aligned}$$

To ensure identification, we impose that the dummy coefficients sum up to zero by classification factors. Substituting these parametric constraints in the model, we obtain that

$$\begin{aligned}
Y_{i,t} = & \rho Y_{i,t-1} + \beta_1 \ln A_{t-1} + \beta_2 \ln e_{i,t-1} + \\
& + \sum_{j=1}^2 \gamma_j (d_{ji} - d_{3i}) \ln A_{t-1} + \sum_{j=1}^2 \gamma'_j (d_{ji} - d_{3i}) \ln e_{i,t-1} \\
& + \sum_{k=1}^3 \alpha_k (D_{ki} - D_{4i}) \ln A_{t-1} + \sum_{k=1}^3 \alpha'_k (D_{ki} - D_{4i}) \ln e_{i,t-1} + (\eta_i + u_{it})
\end{aligned}$$

with $\gamma_3 = -\sum_{j=1}^2 \gamma_j$, $\gamma'_3 = -\sum_{j=1}^2 \gamma'_j$, $\alpha_4 = -\sum_{k=1}^3 \alpha_k$, and $\alpha'_4 = -\sum_{k=1}^3 \alpha'_k$. The final model was estimated in STATA using the Blundell-Bond refinement. We used 6,728 observations with 260 customers, and $1.1e + 03$ instruments. The Wald global significance test is 195.43 distributed as a χ^2_{11} with a p-value 0.0000. The individual marketing effort has a significant impact, as well as the general advertising. The dummy coefficients $\{\gamma'_j\}_{j=1}^2$ are non significant, and set them equal to zero in the optimization part. All the other types of dummy coefficients are significant. After the model coefficients have been estimated, since T is large, we can consistently estimate each specific intercept η_i . For each customer we need to take time-means on the panel regression equations, then replace $\sum_{t=1}^T u_{it}/T$ by zero (the expected value), and finally getting the estimator of η_i .

Next, consider a SDP problem for the returns function

$$r(Y_t, A_t, e_t) = \sum_{i=1}^I Y_{it} - A_t - \sum_{i=1}^I e_{it},$$

where the state variable $\{Y_{it}\}$ are returns drawn from the i -th customer. The transition equations for all customers in one of the identified segments are identical, but there are relevant different across segments.

We have computed the optimal general advertising and marketing effort policies for a stylized version of the model with 12 representative customers, applying the proposed decomposition method. The collocation algorithm was run using a state discretization with 10 scenarios (sales levels, disguised by company request) for each individual-sales variable and 20 equidistant knots for each control, applying policy iteration for each subproblem. It takes 7 iterations (about 11 minutes) of the full decomposition method for the algorithm to converge. Figures 1 and 2 show $\{V_i(s_i)\}_{i=1}^{12}$ and $\bar{V}(\bar{s})$, the individual and mean reduced value functions respectively.

Figure 1: Individual reduced value functions (customer value associated to its sales state).

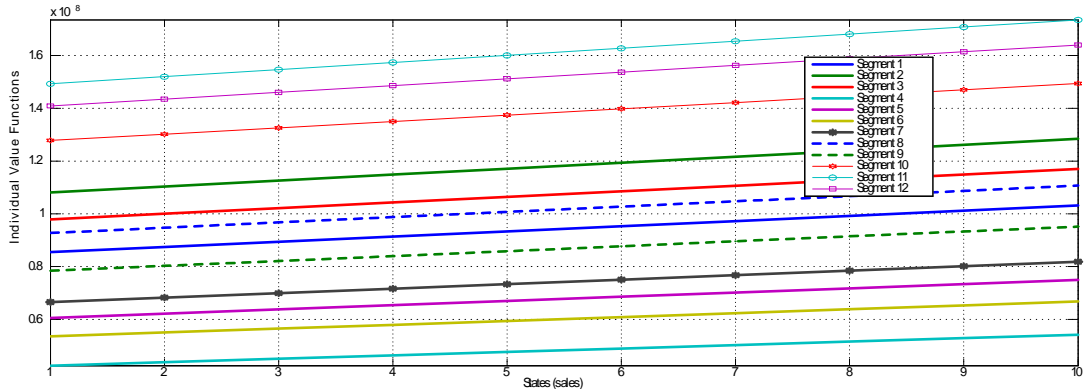
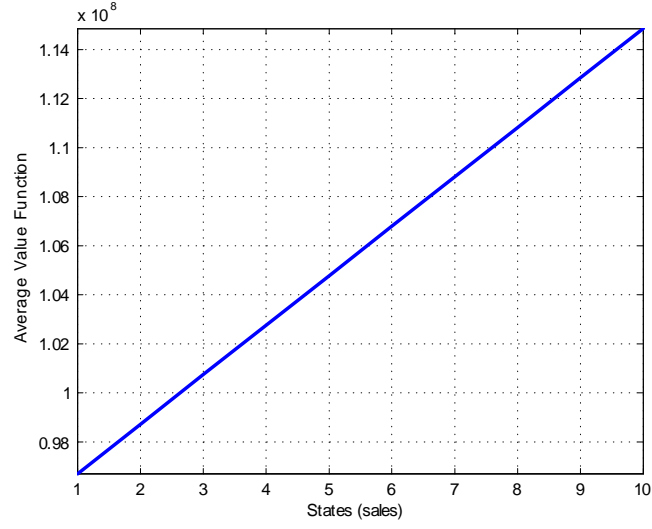


Figure 2: Mean reduced value function (total value associated to mean sales states).



Figures 3 and 4 show $\{e_i(s_i)\}_{i=1}^{12}$ and $A(\bar{s})$, the optimal individual and general marketing effort reduced policy functions respectively.

Figure 3: Individual marketing effort reduced policy functions.

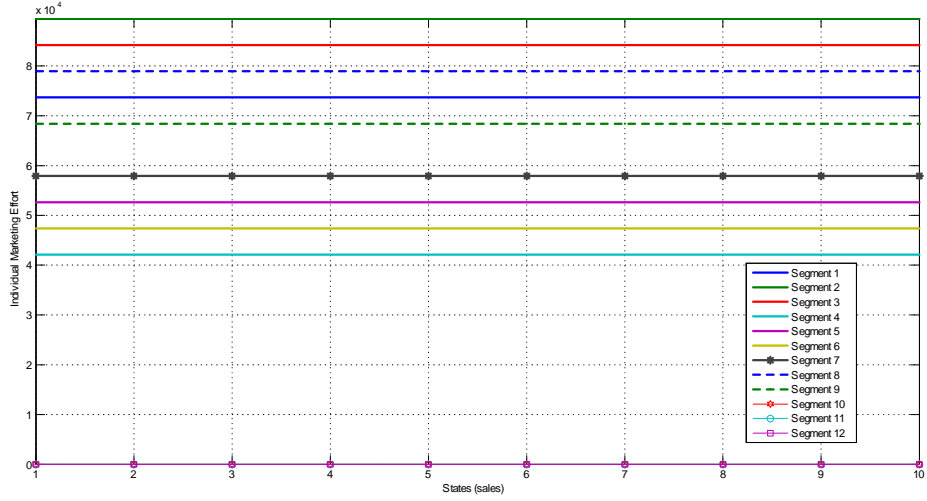
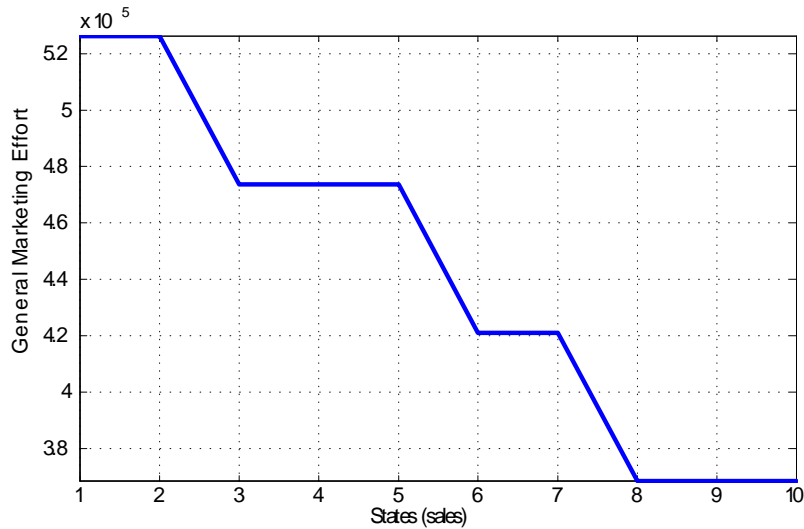


Figure 4: General marketing effort reduced policy functions.



These results show that the optimal budget allocated to mass marketing is decreasing with respect to total sales. By contrast, the individual effort is hold constant with sales but, the level is different for each segment. In particular for the 3 segments which have a negligible individual marketing coefficient, the optimal solution prescribes not to invest at all on them. Furthermore, notice that the ranking of individual effort investments by segments does not follow exactly the pattern given by individual reduced value functions. This is not a surprising result as the optimal solution takes into account not just differences in profitability but also different sensibilities of the segments to the marketing mix.

6 Conclusions

There is a growing interest for firms to customize their marketing activities to smaller and smaller units —individual stores, customers and transactions” (Buckling et al., 1998), implying an enormous number of decisions. This scale requires Decision Automation tools based on dynamic optimization of small unit panels.

In this paper, we make a computational contribution for solving SDP problems, which allows forward-looking firms to allocate the marketing budget optimizing the CLV of their customer base, simultaneously using customized and mass marketing interventions. The solvability of these models suffers from the curse of dimensionality, which limits practitioners from the modelling standpoint. In this sense, we have introduced a novelty decomposition methodology for the computation of solutions of CRM problems. The proposed approach deflates the dimensionality of the models by breaking the problem into a set of smaller independent subproblems. The numerical results have revealed the efficiency of the methodology in terms of computing time and accuracy, concluding that the proposed approach is promising for application in many marketing problems with similar structure.

We have shown the decomposition method works very well in practice. The methodology has been successfully applied to value more than 260 customers of a medium size international wholesale company. We have presented a customer profitability analysis of the company considering the effect of direct marketing and mass marketing interventions at the customer level, simultaneously.

Since often CRM databases do not involve panel data across several competitors, no competitive effects have been considered in this article. To include competition, we should consider a behavioral model for several firms competing for the same customers with mass and customized marketing actions, and the equilibrium would be given by the Markov perfect equilibrium (see Dubé et al. 2005). The computational effort to solve this problem is formidable, and the decomposition algorithm presented in this article could be a useful tool to address it. We leave this problem for future research.

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8 Appendix A: Value iteration and Policy iteration for continuous problems

Continuous SDP problems are usually solved combining the ideas of value iteration and policy iteration with collocation methods. The basic idea of Collocation methods is to consider a sequence of functions $\{\phi_k\}_{k \geq 1} \subset B_\infty$ such that any function $v \in B_\infty$ can be expressed asymptotically as a linear combination of these functions, or more formally for all $v \in B_\infty$

$$\inf_{\{\theta_k\}_{k=1}^K} \left\| v(s) - \sum_{k=1}^K \theta_k \phi_k(s) \right\|_\infty \xrightarrow{K \rightarrow \infty} 0,$$

and therefore we can express $V(s) \approx \sum_{k=1}^K \theta_k \phi_k(s)$ for some coefficients $\{\theta_k\}$ and a large enough K . Several classes of functions that can be used for the approximation (e.g., Chebyshev polynomial, splines, Neural Networks, etc.). When the state variable is multidimensional, the base functions are generally obtained by tensor products on univariate basis. The integer K is exponentially increased with the dimension to obtain a good approximation (this is one type of the *curse of dimensionality*). Notice that the continuous SDP problem can be approximated by another one with finite states (just considering a finite partition $\{\mathcal{S}_k\}$ of the Euclidean state's space \mathcal{S} , we can approximating v by simple functions $\sum_{k=1}^K \theta_k I(s \in \mathcal{S}_k)$, choosing a representative scenario s_k for each element of the partition and interpreting $\theta_k = v(s_k)$).

The coefficients $\{\theta_k\}_{k=1}^K$ are unknown, the collocation method approximates a functional equation in such a way that the approximated function fits exactly at the pre-specified points of the domain. Then, Bellman's Equation becomes

$$\sum_{k=1}^K \theta_k \phi_k(s) = \max_{(A,P,e) \in \mathbb{A}(s)} \left\{ r(s, A, P, e) + \delta \sum_{k=1}^K \theta_k \int \phi_k(s') F(ds'|s, A, P, e) \right\}. \quad (6)$$

Next, we evaluate the linear equation at K grid-points $\{s_1, \dots, s_K\} \subset \mathcal{S}$ and solve the system in $\{\theta_k\}_{k=1}^K$. The system (6) can be expressed in matrix notation as

$$\Phi \theta = \Gamma(\theta) \quad (7)$$

where $K \times K$ matrix Φ has element $\Phi_{mk} = \phi_k(s_m)$ and the $K \times 1$ vector $\Gamma(\theta)$ has m -th element

$$\Gamma_m(\theta) = \max_{(A,P,e) \in \mathbb{A}(s_m)} \left\{ r(s_m, A, P, e) + \delta \sum_{k=1}^K \theta_k \int \phi_k(s') F(ds'|s_m, A, P, e) \right\}$$

The solution of this system is not trivial, first we need to evaluate the expectations

$$\int \phi_k(s') F(ds'|s_m, A, P, e), \quad (8)$$

for $m = 1, \dots, K$; often using a numerical integration method or a Monte Carlo approach. When the integral is replaced by an average over a finite set of sampled points, the number of required points required to have a good approach increases exponentially with the dimension of the state variables (this is another type of *curse of dimensionality*). After computing these expectations, it is generally impossible to attain closed form solution to the collocation system (7), and some computational algorithm is required.

- The *Value iteration* method considers the system $\theta = \Phi^{-1}\Gamma(\theta)$, and iterates the following:

$$\theta \leftarrow \Phi^{-1}\Gamma(\theta)$$

from an initial point θ^0 . It was initially proposed by Bellman (1955, 1957) for discrete problems.

- The *Policy iteration* method uses the Newton iterative updating,

$$\theta \longleftarrow \theta - [\Phi - \Gamma'(\theta)]^{-1} [\Phi\theta - \Gamma(\theta)]$$

where $\Gamma'(\theta)$ is the Jacobian of the collocation function Γ at θ that can be computed *by* applying the *Envelope Theorem* to the optimization problem in the definition of $\Gamma(\theta)$, so that

$$\Gamma'_{mj}(\theta) = \delta \int \phi_j(s') F(ds'|s_m, A, P, e)$$

This method was initially proposed by Howard (1960).

Notice that when the approximation method is based on simple functions, then Φ is the identity function, and we can omit this factor. Each time that the operator $\Gamma(\theta)$ is applied we must solve the maximization problem in $\Gamma_m(\theta)$ for all states $s_m \in \{s_1, \dots, s_K\}$. This can be done, e.g., using a global optimization algorithm. In many applications, the maximization is carried out discretizing the decision space $\mathbb{A}(s_m)$. Once we have converged, $V(s) = \sum_{k=1}^K \theta_k \phi_k(s)$, and the optimal policy is computed at each state $s_m \in \{s_1, \dots, s_K\}$, as the maximizing decision taken at $\Gamma_m(\theta)$ for the last iteration and the function is computed interpolating these points. The main problem with the all previous techniques is the *curse of dimensionality* (Bellman, 1961). So far, researchers can solve numerically only SDP problems with very few state variables.

9 Appendix B: Convergence Analysis

In this section we discuss the convergence of the algorithm. We first introduce some basic notation. The convergence of classical Value Iteration method is based on central ideas from functional analysis. Define the operator

$$\Gamma(v) = \max_{(A, P, e) \in \mathbb{A}(s)} \left\{ r(s, A, P, e) + \delta \int v(s') F(ds'|s, A, P, e) \right\}$$

transforming a function of the state variables $v(s)$ into another function $\Gamma(v)(s)$. Obviously that value function is a fixed point of Γ , i.e. an element v^* such that $\Gamma(v^*) = v^*$. The value iteration algorithm considers an arbitrary function v_0 , and compute recursively $v_j = \Gamma(v_{j-1})$. Under regularity conditions, the sequence $\{v_j\}_{j \geq 1}$ converges to a limit which is the value function v^* .

The argument uses basic concepts of functional analysis. Convergence can be ensured, provided that Γ is a contractive operator in a complete metric space. If B is a complete² metric space, an operator $\Gamma : B \rightarrow B$ is called contractive if $d(\Gamma(v), \Gamma(v')) \leq cd(v, v')$ for all $v, v' \in B$ with parameter $c \in (0, 1)$. Any contractive operator in a complete metric space has a unique fixed point v^* , and satisfies that $v^* = \lim_{j \rightarrow \infty} \Gamma^j(v^0)$ for any initial point $v^0 \in B$, so that the sequence $v^j = \Gamma(v^{j-1}) = \Gamma^j(v^0)$ converges to the fixed point, for an introduction see Kolmogorov and Fomin (1970). In particular we consider the Banach³ space B_∞ of bounded and Borel-measurable real valued functions defined on the Euclidean state's space \mathcal{S} , and endowed with the supremum norm $\|v\|_\infty = \sup_y |v(y)|$. If the function $|r(s, A, P, e)|$ is bounded on \mathcal{K} , then it is easy to prove that $\Gamma(v)$ is a contractive operator on B_∞ with parameter $\delta \in (0, 1)$, and the fixed point $V = \Gamma(V)$ solves the SDP⁴, see e.g. Denardo (1967), and Blackwell (1965). Under stronger conditions on the SDP problem, the value function V can be proved to be continuous, Lipschitz, once/twice continuously differentiable.

Unfortunately, the implementation of the algorithms is unfeasible with more than 3-4 state variables, as the computation of $\Gamma(v)$ requires approximation of the numerical integral $\int v(s') F(ds'|s, A, P, e)$ by an average at selected points, and the number of required points to provide an accurate estimate increases exponentially with the dimension of the state variables.

²A metric space B is complete if it is equal to its closure

³A Banach space is a normed linear space, which is complete with respect to the distance $d(v, v') = \|v - v'\|$ defined from its norm.

⁴There are also extensions for the case where $r(s, A, P, e)$ is bounded on compact subsets, by using other distances (see Rincón-Zapatero and Rodríguez-Palmero, 2003).

Next we discuss the convergence of the presented algorithm. Recall that $V(S_0) = I^{-1} \sum_{i \in \mathbb{I}} V_i(S_{0i}) = \bar{V}(\bar{S}_0)$, where

$$\begin{aligned}\bar{V}(\bar{s}) &= \max_{\{A_t, P_t\}} E_0 \left[\sum_{t \geq 0} \delta^t R(\bar{S}_t, A_t, P_t) \mid \bar{S}_0 = \bar{s} \right], \\ V_i(s_i) &= \max_{\{e_t\}} E_0 \left[\sum_{t \geq 0} \delta^t R_i(S_{it}, e_{it}) \mid S_{i0} = s_i \right].\end{aligned}$$

Consider the operators:

$$\begin{aligned}\Upsilon_i(V_i, A, P)(s_i) &= \max_{\{e_i \in \mathbb{A}_i(s_i)\}} \left\{ R_i(S_{it}, e_{it}) + \delta \int V_i(s'_i) F^{A, P}(s'_i \mid s_i, e_i) \right\}, \\ \Phi(\bar{V}, e)(\bar{s}) &= \max_{\{A, P \in \bar{\mathbb{A}}(\bar{s})\}} \left\{ R(\bar{S}_t, A_t, P_t) + \delta \int \bar{V}(\bar{s}') F^e(d\bar{s}' \mid \bar{s}, A, P) \right\}.\end{aligned}$$

where $F^{A, P}(s'_i \mid s_i, e_i)$, $F^e(d\bar{s}' \mid \bar{s}, A, P)$ are defined as in the algorithm steps (2.1) and (2.3). The arguments that maximize these two problems are $\{e_i(s_i)\}_{i=1}^I$ and $(A(\bar{s}), P(\bar{s}))$, respectively. The convergence of the decomposition algorithm can be deduced similarly to the proof of convergence of the policy iteration method, using the following arguments:

1) The solution to the functional equation system

$$\begin{aligned}\Upsilon_i(V_i, A, P)(s_i) &= V_i(s_i), \quad i = 1, \dots, n \\ \Phi(\bar{V}, e)(\bar{s}) &= \bar{V}(\bar{s})\end{aligned}$$

satisfies by construction that $V(s) = I^{-1} \sum_{i=1}^I V_i(s_i, A(\bar{s}), P(\bar{s})) = \bar{V}(\bar{s}, \{e_i(s_i)\})$ a.e., where $V(s)$ is the value function of the original SDP problem.

2) The algorithm can be considered as a recursion defined by a contractive operator. Consider some initial value $V(s) \in B_\infty$, then we can write $V = \frac{1}{I} \sum_{i=1}^I V_i$ for a vector (V_1, \dots, V_I) with coordinates $V_i = \Pi_i V(s)$, where the operator Π_i is defined as:

$$\begin{aligned}\Pi_i v(s) &= E \left[\sum_{t \geq 0} \delta^t R_{iv}(S_{it}, e_{iv}(S_{it})) \mid S_{i0} = s_i \right], \\ R_{iv}(S_{it}, e_{iv}(S_{it})) &= E[I \cdot r_i(S_{it}, e_{iv}(S_{it}), P_v(S_t), A_v(S_{it})) \mid S_{it}]\end{aligned}$$

and $A_v(s)$, $P_v(s)$, $e_v(s)$ are the policies rendering the value function $v(s)$. These operators satisfy $\|\Pi_i(v)\|_\infty \leq \|v\|_\infty$.

The algorithm can be regarded as a sequence obtained alternating the operators $(\beta_1, \dots, \beta_I)$ from $B_\infty \rightarrow B_\infty$ defined by $\beta_i = \Upsilon_i \circ \Pi_i V$, with the operator Φ . In other words, it is a recursion defined by the operator $\Delta = \left(\Phi \circ \frac{1}{I} \sum_{i=1}^I \beta_i \right)$ from $B_\infty \rightarrow B_\infty$. The operator Δ is a contractive operator on B_∞ , since Φ and Υ_i are Bellman operators (contractive with parameter δ),

$$\begin{aligned}\|\Delta(v)\|_\infty &= \left\| \Phi \circ \left(\frac{1}{I} \sum_{i=1}^I \beta_i \right) (v) \right\|_\infty \leq \delta \left\| \frac{1}{I} \sum_{i=1}^I \beta_i(v) \right\|_\infty \leq \delta \frac{1}{I} \sum_{i=1}^I \|\Upsilon_i \circ \Pi_i(v)\|_\infty \\ &\leq \delta^2 \frac{1}{I} \sum_{i=1}^I \|\Pi_i(v)\|_\infty \leq \delta^2 \|v\|_\infty\end{aligned}$$

and we can apply a fixed point theorem to the alternating operator Δ to prove convergence to a fixed point satisfying the conditions in 1).

10 Appendix C: Algorithm Implementation

The first step follows the discretization technique. Mainly, we consider a grid of controls, $\{\mathbf{A}, \mathbf{P}, \mathbf{e}_1, \dots, \mathbf{e}_I\}$, containing a discretization of the feasible decision set. In particular we consider relatively large finite intervals for each decision, and introduce N equidistant points for each decision.

The second step is the definition of the scenario nodes and transition probabilities across scenario states. The unconditional distribution can be used to define a grid of representative state values, and the conditional distribution to compute the transition matrix across the elements of the grid. In particular, when we consider the model 1 $S_{it} = \rho S_{it-1} + g_i + \varepsilon_{it}$ where $\varepsilon_{it} = \eta_i + u_{it} \sim N(0, \sigma_{\varepsilon_i}^2)$ with $\sigma_{\varepsilon}^2 = \sigma_{\eta}^2 + \sigma_u^2$, and

$$\begin{aligned} S_{it}|S_{it-1}, A, P, e &\sim N(\rho S_{it-1} + g_i, \sigma_{\varepsilon}^2), \\ \bar{S}_{It}|\bar{S}_{It-1}, A, P, e &\sim N\left(\rho \bar{S}_{It-1} + \bar{g}, \frac{\sigma_{\varepsilon}^2}{I}\right), \end{aligned}$$

with $g_i = g_i(A, P, e_i)$, $\bar{g}(A, P, e) = \sum_i g_i(A, P, e_i)/I$. The stationary marginal distribution of S_{ti} and \bar{S}_t are $N\left(\frac{g_i(A, P, e_i)}{(1-\rho)}, \frac{\sigma_{\varepsilon}^2}{(1-\rho^2)}\right)$ and $N\left(\frac{\bar{g}(A, P, e)}{(1-\rho)}, \frac{\sigma_{\varepsilon}^2}{I \cdot (1-\rho^2)}\right)$, respectively. For the i -th customer, we set scenarios in the interval $[S_i^l, S_i^u]$, where

$$\begin{aligned} S_i^l &= \min_{A, P, e_i} \frac{g_i(A, P, e_i)}{(1-\rho)} - 5\sqrt{\frac{\sigma_{\varepsilon}^2}{(1-\rho^2)}}, \\ S_i^u &= \max_{A, P, e_i} \frac{g_i(A, P, e_i)}{(1-\rho)} + 5\sqrt{\frac{\sigma_{\varepsilon}^2}{(1-\rho^2)}} \end{aligned}$$

Therefore, we cover 5 times the standard deviation from the most extreme mean values. After checking that $\max\{S_i^l, 0\} < S_i^u$ we generate N scenarios distributed uniformly as

$$\begin{aligned} s_{i1} &= \max\{S_i^l, 0\}, \\ s_{iN} &= S_i^u, \\ s_{in} &= s_{i1} + \left(\frac{s_{iN} - s_{i1}}{N-1}\right)(n-1), \quad n = 2, 3, \dots, N-1. \end{aligned}$$

Then we define the product space of states $\mathcal{S}^I = \prod_{i=1}^I \{s_{i1}, \dots, s_{iN}\}$. The discrete scenario grid \mathcal{S}^I be used to compute the Bellman problem, defining the value functions and the policy functions as mappings defined on \mathcal{S}^I .

However, in our context it is convenient to think of an augmented space of states including mean sales. Consider the mean interval $[S^l, S^u]$, with $S^l = \sum_{i \in \mathbb{I}} S_i^l/I$ and $S^u = \sum_{i \in \mathbb{I}} S_i^u/I$, and generate N scenarios $\{\bar{s}_1, \dots, \bar{s}_N\}$ distributed uniformly in $\max\{S^l, 0\} < S^u$. Therefore, we can define the augmented space as

$$\mathcal{S}^{I+1} = \left\{ (s, \bar{s}) : s = (s_1, \dots, s_I)' \in \mathcal{S}^I, \bar{s} \simeq \frac{1}{I} \sum_{i \in \mathbb{I}} s_i \right\},$$

where \simeq means that \bar{s} is the scenario in $\{\bar{s}_1, \dots, \bar{s}_N\}$ closest to $\sum_{i \in \mathbb{I}} s_i/I$. Thus a specific realization of the random vector (S_t, \bar{S}_t) will be approached by a vector $(s, \bar{s}) \in \mathcal{S}^{I+1}$. Given the structure of the problem, we can define the policy functions (A^k, P^k, e^k) in the augmented space as a mapping

$$(A^k, P^k, e^k) : \mathcal{S}^{I+1} \ni s \rightarrow (A^k(\bar{s}), P^k(\bar{s}), e_1^k(s_1), \dots, e_I^k(s_I)) \in \{\mathbf{A}, \mathbf{P}, \mathbf{e}_1, \dots, \mathbf{e}_I\}.$$

The value function can be approximated in \mathcal{S}^{I+1} by a simple function,

$$v(s, \bar{s}) = \sum_{n_1, \dots, n_I, n_{I+1}} \theta_{n_1, \dots, n_I, n_{I+1}} \cdot \left\{ \prod_{i=1}^I I(b_{n_i-1} < s_i \leq b_{n_i}) \cdot I(b_{n_{I+1}-1} < \bar{s} \leq b_{n_{I+1}}) \right\}.$$

An smooth functional basis could be considered instead of simple functions, e.g. replacing the bracket in the previous expression by a tensor product of orthonormal polynomials.

We need to compute $\mathbb{F}^k(s'_i|s_i, e_i)$ and $\mathbb{F}^k(\bar{s}'|\bar{s}, A, P)$ in Step 2.2. In order to marginalize the effect of some policy controls over the transition probabilities, we apply the Monte Carlo method. First, given the policy (A^k, P^k, e^k) we generate recursively a sample $\{S_t^k, A_t^k, P_t^k, e_t^k\}_{t=1}^T$ as

$$\begin{aligned} S_{it}^k &= \rho S_{it-1}^k + g_i(A_{t-1}^k, P_{t-1}^k, e_{it-1}^k) + \varepsilon_{it}, & i \in \mathbb{I} \\ \bar{S}_{t-1}^k &= I^{-1} \sum_{i \in \mathbb{I}} S_{it}^k \end{aligned}$$

with $\varepsilon_i \sim N(0, \sigma_{\varepsilon_i}^2 I_T)$ and $S_{i0}^k = 0$, and compute recursively the associated controls as follows:

$$\begin{aligned} A_t^k &= \sum_{n=1}^N A^k(\bar{s}_n) I(b_{n-1} < \bar{S}_{t-1}^k \leq b_n) \\ P_t^k &= \sum_{n=1}^N P^k(\bar{s}_n) I(b_{n-1} < \bar{S}_{t-1}^k \leq b_n) \\ e_{it}^k &= \sum_{n=1}^N e_i^k(s_{in}) I(b_{i,n-1} < S_{i,t-1}^k \leq b_{i,n}), & i \in \mathbb{I}, \end{aligned}$$

where $b_n = (\bar{s}_{n+1} + \bar{s}_n)/2$ and $b_{i,n-1} = (s_{i,n+1} + s_{i,n})/2$ for $n = 1, \dots, N-1$, and we set $b_0 = b_{i,0} = -\infty$ and $b_N = b_{i,N} = +\infty$. The last expressions are used due to the fact that the policy functions are defined for discrete scenarios, for example we set $A_t^k = A^k(\bar{s}_n)$ whenever $\bar{S}_{t-1}^k \in (b_{n-1}, b_n]$ which is the interval centered in \bar{s}_n . We throw away the first 100 observations to remove the effect of the initial data, and continue to generate a large sample with at least $T = 3000$ observations, but this figure could be doubled when the diameter of the feasible decision set or N increases.

In order to define properly the objective function for each subproblem, we compute certain conditional expectations and transition kernels using the simulated sample $\{S_t^k, A_t^k, P_t^k, e_t^k\}_{t=1}^T$. First, for all $i \in \mathbb{I}$ we compute the conditional expectations $P_{in}^k = E[P_t^k | S_{it}^k = s_{in}]$, $C_{in}^k = E[c_m(A_t^k) | S_{it}^k = s_{in}]$, at the discrete scenarios $\{s_{in}\}_{n=1}^N$ and $c_{in}^k = E[c_i(e_{it}^k) | \bar{S}_t^k = \bar{s}_n]$ at the scenarios $\{\bar{s}_n\}_{n=1}^N$. Then we compute an approximation of the subproblem objective functions (5) evaluated at the discrete scenarios as

$$\begin{aligned} R_i^k(s_{in}, e_{it}) &= I \cdot ((P_{in}^k - c_0) \cdot s_{in} - c_i(e_{it}) - I^{-1} C_{in}^k), \\ R^k(\bar{s}_n, A_t, P_t) &= (P_t - c_0) \cdot I \cdot \bar{s}_n - \sum_{i \in \mathbb{I}} c_{in}^k - c_m(A_t). \end{aligned}$$

The fastest method to compute the conditional expectations is based on a simple parametric regression model (e.g., specifying $E[P_t^k | S_{it}^k = s_i] = p(s_i, \beta)$). The model is estimated by a least squares method (e.g., minimizing $\sum_{t=1}^T (P_t^k - p(S_{it}^k, \beta))^2$) for direct use (setting $P_{in}^k = p(s_{in}, \hat{\beta}^K)$ for each discrete scenario s_{in}). The parametric approach works well in our application. Alternatively we can use a nonparametric estimator. For example the Nadaraya-Watson estimator of $E[P_t^k | S_{it}^k = s_{in}]$, is given by

$$E[P_t^k | S_{it}^k = s_{in}] = \frac{\sum_{t=1}^T P_t^k K_{h_T}(S_{it}^k - s_{in})}{\sum_{t=1}^T K_{h_T}(S_{it}^k - s_{in})}$$

where $K_{h_T}(u) = h_T^{-1} K(u/h_T)$ for an arbitrary kernel density $K(\cdot)$ (e.g. a standard normal density), and a sequence of positive smoothing parameters h_T such that $h_T + (Th_T)^{-1} \rightarrow 0$. This approach avoids specification assumptions, but it requires larger sample sizes T than the parametric approach. Besides, an optimal selection of the smoothing parameter is crucial, which is time consuming. However, it might be convenient in some applications.

Second we compute the marginal transition kernels $\mathbb{F}^k(s'_i|s_i, e_i)$ and $\mathbb{F}^k(\bar{s}'|\bar{s}, A, P)$. There are several possibilities: parametric methods, semiparametric, and nonparametric. The fastest method is based on a parametric model, postulating regression model, $E[S_{it}^k|S_{it-1}^k, e_{it-1}^k] = m_i(S_{it-1}^k, e_{it-1}^k, \beta_i)$, $E[\bar{S}_t^k|\bar{S}_{t-1}^k, A_{t-1}^k, P_{t-1}^k] = \bar{m}(\bar{S}_{t-1}^k, e_{it-1}^k, \theta)$, estimating the model by a ordinary/nonlinear least squares method. In our applications we consider this method for a linear in parameters model without intercept where first regressor is in levels and the controls are in logarithms. Assume that the errors are conditionally independent of the state variables, we can use the residuals

$$\begin{aligned}\hat{u}_{it} &= S_{it}^k - m_i(S_{it-1}^k, e_{it-1}^k, \hat{\beta}_i) \\ \hat{\bar{u}}_t &= \bar{S}_t^k - \bar{m}(\bar{S}_{t-1}^k, A_{t-1}^k, P_{t-1}^k, \hat{\theta})\end{aligned}$$

to estimate the error densities $g_i(u_{it})$, $\bar{g}(\bar{u}_t)$. In particular we have assumed Gaussian distributions $N(0, \sigma_{u_i}^2)$ and $N(0, \sigma_{\bar{u}}^2)$ respectively, estimating the variances $\sigma_{u_i}^2$ and $\sigma_{\bar{u}}^2$ with the mean squared residuals, we get

$$\begin{aligned}\mathbb{F}_i(s'_i|s_i, e_i) &= \frac{1}{\hat{\sigma}_{u_i}} \int_{-\infty}^{s'_i} \phi\left(\frac{z - m_i(s_i, e_i, \hat{\beta}_i)}{\hat{\sigma}_{u_i}}\right) dz = \Phi\left(\frac{s'_i - m_i(s_i, e_i, \hat{\beta}_i)}{\hat{\sigma}_{u_i}}\right), \\ \mathbb{F}(\bar{s}'|\bar{s}, A, P) &= \frac{1}{\hat{\sigma}_{\bar{u}}} \int_{-\infty}^{\bar{s}'} \phi\left(\frac{z - \bar{m}(\bar{s}, A, P, \hat{\theta})}{\hat{\sigma}_{\bar{u}}}\right) dz = \Phi\left(\frac{\hat{\bar{u}}_t - (\bar{s}' - \bar{m}(\bar{s}, A, P, \hat{\theta}))}{\hat{\sigma}_{\bar{u}}}\right)\end{aligned}$$

Notice that if it is difficult to determine the residuals distribution, we could estimate $g_i(u_{it})$, $\bar{g}(\bar{u}_t)$ nonparametrically. For example, integrating the Rosenblatt-Parzen kernel density estimator we obtain a cumulative conditional distribution

$$\begin{aligned}\mathbb{F}_i(s'_i|s_i, e_i) &= \int_{-\infty}^{s'_i} \left(\frac{1}{T-2} \sum_{t=2}^T K_{h_T}(\hat{u}_{it} - (z - m_i(s_i, e_i, \hat{\beta}_i))) \right) dz, \\ \mathbb{F}(\bar{s}'|\bar{s}, A, P) &= \int_{-\infty}^{\bar{s}'} \left(\frac{1}{T-2} \sum_{t=2}^T K_{h_T}(\hat{\bar{u}}_t - (z - \bar{m}(\bar{s}, A, P, \hat{\theta}))) \right) dz,\end{aligned}$$

where $K_{h_T}(u) = h_T^{-1}K(u/h_T)$. This semiparametric method slows down the algorithm compared with the parametric case. The last alternative is a fully nonparametric estimator such as the cumulated integral of the conditional density estimator by Roussas (1967, 1969) and Chen, Linton and Robinson (2001),

$$\begin{aligned}\mathbb{F}_i(s'_i|s_i, e_i) &= \int_{-\infty}^{s'_i} \frac{\sum_{t=2}^T K_{h_T}(S_{it}^k - z) K_{h_T}(S_{it-1}^k - s_i) K_{h_T}(e_{it-1}^k - e_i)}{\sum_{t=2}^T K_{h_T}(S_{it-1}^k - s_i) K_{h_T}(e_{it-1}^k - e_i)} dz \\ \mathbb{F}(\bar{s}'|\bar{s}, A, P) &= \int_{-\infty}^{\bar{s}'} \frac{\sum_{t=2}^T K_{h_T}(\bar{S}_t^k - z) K_{h_T}(\bar{S}_{t-1}^k - \bar{s}) K_{h_T}(A_{t-1}^k - A) K_{h_T}(P_{t-1}^k - P)}{\sum_{t=2}^T K_{h_T}(\bar{S}_{t-1}^k - \bar{s}) K_{h_T}(A_{t-1}^k - A) K_{h_T}(P_{t-1}^k - P)} dz\end{aligned}$$

This method requires very large simulated samples, and it is quite sensitive to the selection of the smoothing number that must be optimally determined. In general we do not recommend it for this algorithm, but it might be useful in some applications.

To apply the collocation method for the Bellman equation associated to each subproblem we have to integrate the basis functions with respect to $\mathbb{F}_i(s'_i|s_i, e_i)$ and $\mathbb{F}(\bar{s}'|\bar{s}, A, P)$, which requires a numerical integration method. We use the Tauchen's method (1986) to approximate the continuous transition kernel $\mathbb{F}^k(s'_i|s_i, e_i)$ and $\mathbb{F}^k(\bar{s}'|\bar{s}, A, P)$ by analogous finite-state transition matrix on the states grid $\{s_1, \dots, s_N\}$, considering for all $n, m = 1, \dots, N$ the transition from s_n to s_m

$$\begin{aligned}\Pi_{nm}^i(e_i) &= \mathbb{F}_i(b_{i,m}|s_{in}, e_i) - \mathbb{F}_i(b_{i,m-1}|s_{in}, e_i), \\ \Pi_{nm}^{mean}(A, P) &= \mathbb{F}(b_m|s_{20}, A, P) - \mathbb{F}(b_{m-1}|s_n, A, P),\end{aligned}$$

where $b_{i,m} = (s_{i,m+1} + s_{i,m})/2$, $b_m = (\bar{s}_{m+1} + \bar{s}_m)/2$ for $m = 1, \dots, N-1$, and we set $b_{i,0} = b_0 = -\infty$ and $b_{i,N} = b_N = +\infty$ so that $\Pi_{n1}^i(A, P, e) = \mathbb{F}_i(b_1|s_n, e_i)$, $\Pi_{nN}^i(A, P, e) = 1 - \mathbb{F}(b_{N-1}|s_n, A, P)$, and similarly for $\Pi_{n1}^{mean}(A, P)$ and $\Pi_{nN}^{mean}(A, P)$. In order to apply the collocation value iteration, or policy iteration method, the continuous-state expectations of the basis functions (8) for each subproblem, namely $\int \phi_k(s') \mathbb{F}_i(ds'|s_m, e_i)$ and $\int \phi_k(s') \mathbb{F}(ds'|s_m, A, P)$, are approximated by the expected values in the analogous discrete Markov chain $N^{-1} \sum_{n=1}^N \phi_k(s'_n) \Pi_{nm}^i(e_i)$ and $N^{-1} \sum_{n=1}^N \phi_k(s'_n) \Pi_{nm}^{mean}(A, P)$ respectively.